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ON THE REPRESENTATION OF A NUMBER AS THE SUM OF ANY NUMBER OF SQUARES, AND IN PARTICULAR OF FIVE OR SEVEN

By G. H. Hardy
Trinity College, Cambridge, England

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1. The formulae concerning the representation of a number as the sum of 5 or 7 squares belong to one of the most unfamiliar and difficult chapters in the Theory of Numbers, and only one proof of them has been given. The proof depends on the general arithmetic theory of quadratic forms, initiated by Eisenstein and perfected by Smith and Minkowski. This theory, of which a systematic account will be found in the fourth volume of Bachmann's *Zahlentheorie* gives a complete solution of the problem of any number s of squares not exceeding 8. Beyond $s = 8$ it fails.

When s is *even* there is an alternative method. This method, which depends on the theory of the elliptic modular functions, is much simpler in idea than the method of Smith and Minkowski; and it has another very important merit, that it can be used—within the limits of human capacity for calculation—for *any* even value of s. Thus Jacobi solved the problem for 2, 4, 6 and 8. In these cases the number of representations can be expressed in terms of the divisors of n. Suppose, *e.g.*, that $s = 8$; and let us write, generally,

$$1 + \sum_1^\infty r_s(n) q^n = (1 + 2q + 2q^4 + \ldots)^s = \{\vartheta_3(0,\tau)\}^s = \vartheta^s,$$

where $q = e^{\pi i \tau}$. Then

$$\vartheta^8 = 1 + 16\left(\frac{1^3 q}{1+q} + \frac{2^3 q^2}{1-q^2} + \frac{3^3 q^3}{1+q^3} + \frac{4^3 q^4}{1-q^4} + \ldots\right),$$

and $r_8(n)$ is $16 \Sigma \delta^3$ if n is odd and $8 \Sigma \delta_0^3 - 8 \Sigma \delta_1^3$ if n is even, δ denoting

a divisor of n, δ_0 a even, and δ_1 an odd divisor. When s exceeds 8 the formulae are less simple, and involve arithmetical functions of a more recondite nature. Liouville gave formulae concerning the cases $s = 10$ and $s = 12$, and Glaisher[1] has worked out systematically all cases up to $s = 18$. More recently important papers on the subject, to which I shall refer later, have been published by Ramanujan[2] and Mordell.[3] In the latter paper the whole subject is exhibited as a corollary of the general theory of modular invariants.

The primary object of my own researches has been *to deduce the formulae for $s = 5$ and $s = 7$ from the theory of elliptic functions*, and so to place the cases in which s is odd and even, so far as may be, on the same footing. The methods which I use have further important applications, but this is the one which I wish to emphasize at the moment. The formulae which I take as my goal are the formulae

$$r_5(n) = \frac{Bn\sqrt{n}}{\pi^2} \sum \left(\frac{n}{m}\right)\frac{1}{m^2}, \tag{1}$$

$$r_7(n) = \frac{Cn^2\sqrt{n}}{\pi^3} \sum \left(\frac{-n}{m}\right)\frac{1}{m^3}, \tag{2}$$

given by Bachmann (pp. 621, 655). Here n as an odd number not divisible by any square (so that there is no distinction between primitive and imprimitive representations); m runs through all odd numbers prime to n; B is 80, 160, 112, or 160, according as n is congruent to 1, 3, 5 or 7 (mod. 8); and C is 448, 560, 448 or 592 in similar circumstances. These formulae are the central formulae of the theory: they involve infinite series, but these series are readily summed in finite terms by the methods of Dirichlet and Cauchy. With them should be associated the formula

$$r_3(n) = \frac{A\sqrt{n}}{\pi} \sum \left(\frac{-n}{m}\right)\frac{1}{m}, \tag{3}$$

where A is 24, 16, 24, or 0: but this formula, as we shall see, stands in some ways on a different footing.

2. My new proof of the formulae (1) and (2) was arrived at incidentally in the course of researches undertaken with a different end, that of finding *asymptotic* formulae (valid for *all* values of s) for $r_s(n)$ and other arithmetical functions which present themselves as coefficients in the expansions of elliptic modular functions. In a paper[4] shortly to appear in the *Proceedings* of the London Mathematical Society, Mr. Ramanujan and I have developed an exceedingly powerful method for the solution of problems of this character, and applied it to the study of $p(n)$, the number of (unrestricted) partitions of n. This method, when applied to our present problem, introduces the function

$$\Theta_s(q) = 1 + \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} \sum_{h,k} \left(\frac{S_{h,k}}{k}\right)^s F(qe^{-h\pi i/k}), \tag{4}$$

where

$$S_{h,k} = \sum_{p=0}^{k-1} e^{p^2 h\pi i/k}, \quad F(q) = \sum_1^\infty n^{\frac{1}{2}s-1} q^n,$$

and the summation applies to $k = 1, 2, 3, \ldots$, and all positive values of h less than, of opposite parity to, and prime to k ($h = 0$ being associated with $k = 1$ alone). The coefficient of q^n in $\Theta_s(q)$ is

$$\chi_s(n) = \frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} n^{\frac{1}{2}s-1} \sum_{h,k} \left(\frac{S_{h,k}}{k}\right)^s e^{-nh\pi i/k}; \tag{5}$$

and our method leads to the conclusion that

$$r_s(n) = \chi_s(n) + O(n^{\frac{1}{2}s}), \tag{6}$$

at any rate for every value of s exceeding 4.

When s is *even*, $F(q)$ is an elementary function; and $(S_{h,k})^s$ is easily expressible in a form which does not involve the 'Legendre-Jacobi symbol' $\left(\frac{a}{b}\right)$. The function $X_s(n)$ is then susceptible of a variety of elementary transformations and it was shown by Ramanujan, in the second of his two papers quoted above, that $X_s(n)$ is *identical with* $r_s(n)$ when $s = 4$, 6 or 8. In what follows I confine myself to the case in which s is *odd*, merely remarking that my method (which is entirely unlike that used by Ramanujan) leads directly to an alternative proof of his results.

3. When s is odd, $F(q)$ is not an elementary function. But it is not difficult to prove that

$$\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)} F(q) = \sum_{-\infty}^{\infty} \frac{1}{[(2n-\tau)i]^{\frac{1}{2}s}}, \tag{7}$$

every term on the right hand side having an argument numerically less than $\frac{1}{4}s\pi$. Further, $S_{h,k}^s = S_{h,k}^{s-1} S_{h,k}$; and the first factor can always be expressed in a simple form. Suppose, to fix our ideas, that $s = 5$. Then $S_{h,k}^4 = (-1)^h k^2$. Substituting from this equation and from (7) into (4), and effecting some obvious simplications, we obtain

$$\Theta_5(q) = 1 + \sum_{h,k} \frac{(-1)^h S_{h,k}}{\sqrt{k}} \frac{1}{[(h-k\tau)i]^{\frac{5}{2}}} \tag{8}$$

where now h assumes all values of opposite parity to and prime to k. This formula may be simplified further by multiplying each side by

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

We then find

$$\Theta_5(q)=\frac{8}{\pi^2}\sum_{h,k}\frac{(-1)^h S_{h,k}}{\sqrt{k}}\frac{1}{[(h-k\tau)i]^{\frac{5}{2}}}, \tag{9}$$

the summation now extending to $k = 0, 1, 2, \ldots$ and *all* h of opposite parity to k. This is our fundamental formula, when $s = 5$. Two steps remain: first, to prove the identity of $\Theta_5(q)$ and ϑ^5; secondly, to deduce the formulae of Smith and Minkowski.

4. The first step presents no very serious difficulty, for it involves nothing beyond an adaptation of the ideas used by Mordell in his paper quoted in §1. We prove first that Θ_5 behaves like ϑ^5 in respect to the linear modular transformations $\tau = T + 2$, $\tau = -1/T$; so that Θ_5/ϑ^5 is an invariant of the modular sub-group called by Klein-Fricke and Mordell Γ_3. Secondly, by studying the transformation $\tau = (T - 1)/T$, we prove that Θ_5/ϑ^5 is *bounded* in the 'fundamental polygon' associated with Γ_3. It then follows that the quotient is a constant which is easily seen to be unity. In all this the only difficulty arises from the use of certain reciprocity-formulae satisfied by Gauss's sums.

We now transform (9) by effecting the summations with respect to h, using certain contour integrals of a type common in the work of Lindelöf and other writers. We thus obtain

$$\vartheta^5=1+\frac{32}{3}\left\{\sum_{1,3,5,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{3}{2}}q^{mk+j}\right. \tag{10}$$

$$\left.-\sum_{2,4,6,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{\frac{3}{2}}q^{mk+j}\right\}$$

a fundamental identity which contains the whole theory of the representation of numbers by sums of 5 squares. The symbols j and μ alone require explanation; j runs through the complete set of least positive residues of $0, 1^2, 2^2, \ldots, (k-1)^2$ to modulus k, each taken as often as it occurs; and μk is the multiple of k deducted in order to arrive at such a residue. And the remainder of the work is purely arithmetical. Picking out the coefficient of q^n, we obtain a series for $r_5(n)$ which is found, after some reduction, to be equivalent to the series given by Bachmann.

4. The formulae which correspond to (10) for $s = 7$ and $s = 3$ are

$$\vartheta^7 = 1 + \frac{256}{15}\left\{\sum_{1,3,5,\ldots}\frac{(-1)^{\frac{1}{2}(k-1)}}{k^3}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{7}{2}}q^{mk+j}\right. \tag{11}$$
$$\left. - \sum_{2,4,6,\ldots}\frac{1}{k^2}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{\frac{7}{2}}q^{mk+j}\right\},$$

$$\vartheta^3 = 1 + 8\left\{\sum_{1,3,5,\ldots}\frac{(-1)^{\frac{1}{2}(k-1)}}{k}\sum_j\sum_{m=0}^{\infty}(mk+j)^{\frac{1}{2}}q^{mk+j}\right. \tag{12}$$
$$\left. + \sum_{2,4,6,\ldots}\frac{1}{k}\sum_j\sum_{m=0}^{\infty}(-1)^{m+\mu}(mk+j)^{\frac{1}{2}}q^{mk+j}\right\}.$$

The interpretation of j and μ is as before, except that, when k is even, j is a residue of one of the numbers $\frac{1}{2}k$, $\frac{1}{2}k + 1^2$, . . . , $\frac{1}{2}k + (k-1)^2$. These identities embody the theory for 7 or 3 squares. It should be noted however, that the application of my method becomes very much more difficult when $s = 3$, as the double series used are then not absolutely convergent; and in fact the only proof of (12) which I possess consists in an identification of the results which it gives with those already known.

I conclude by a word concerning the cases in which $s > 8$. Here, when s is odd, we are on untrodden ground. We have the asymptotic formula (6); and we can evaluate $X_s(n)$ as when $s = 5$ or 7, thus obtaining a series of new results. But it is no longer to be expected that our results should be *exact*, and I have verified that, when $s = 9$, they are not exact, even when $n = 1$.

[1] Glaisher, J. W. L., *Proc. London Math. Soc.*, (Ser. 2), **5**, 1907, (479–490).

[2] Ramanujan, S., *Trans. Camb. Phil. Soc.*, **22**, 1916, (159–184); *Ibid.*, (in course of publication).

[3] Mordell, L. J., *Quart. J. Math.*, **48**, 1917, (93–104).

[4] Hardy, G. H., and Ramanujan, S., *Proc. London Math. Soc.*, (Ser. 2), **17**, 1918, (in course of publication).

THE CRYSTAL STRUCTURE OF ICE

By Ancel St. John

Department of Physics, Lake Forest College

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During the winter of 1916–1917 the crystal structure of ice was investigated by means of the X-rays. The photographic method originated by deBroglie[1] was used with certain modifications suggested privately by Dr. A. W. Hull. The source of energy was a Coolidge tube with tungsten target excited by an induction coil with mercury turbine interrupter. At first the

apparatus already set up for another investigation was used, being kept cool by leaving the laboratory windows open. This method was very uncertain on account of the erratic weather and was otherwise unsatisfactory and was discarded after a single good photograph had been obtained. The spectrometer system was then enclosed and the chamber kept cool by cans of ice and salt. By this means the temperature could be kept reasonably constant but it was found virtually impossible to mount and maintain a specimen long enough to get a satisfactory photograph, probably on account of the presence of the salt vapor. Upon the recommendation of Prof. A. G. Webster a grant was made from the Rumford Fund of the American Academy of Arts and Sciences, Boston, in aid of the investigation which made it possible to install a small ammonia refrigerating machine loaned by the Automatic Refrigerating Company and to build a specially adapted spectrometer mounted in a well-insulated refrigerator box. With this equipment the temperature could be maintained indefinitely and there was no further trouble from melting of specimens. A marked tendency to sublimation, however, was troublesome until each specimen was mounted in a gelatine capsule when equilibrium was quickly established between the crystal and its vapor. Protected in this manner specimens were preserved for days.

Commercial artificial ice was first investigated as it shows marked prismatic structure. Unfortunately the prisms are distorted through pressure in the formation of the ice so that it is difficult to identify cleavage planes. Some photographs show spectral bands but no sharp lines upon which to base calculations. The investigation is to be pursued further, however, as it is probable that sufficiently small crystals or slices of crystals will give sharply defined lines. In a second procedure a thin layer of ice, about 2 mm. thick, was allowed to form on a pan of tap water. It was difficult to identify individual crystals and more difficult to isolate them for mounting but occasionally a reasonably good specimen was secured and mounted so that the axis of rotation bore a definite relation to the original surface of the ice. Orientation with respect to other axes was a matter of guess work, usually wrong as the results showed. Nevertheless several satisfactory photographs were obtained from specimens prepared in this manner, in fact the calculations are based entirely on them. In a third method ice was frozen out of a weak salt solution. In this way a large crop of thin specimens showing distinct cleavage planes could be secured. Owing to the failure of the source of power the investigation was interrupted at this point and has not yet been renewed. One photograph was obtained showing distinct spectral bands but no identifiable lines, possibly on account of the mixture of microscopic salt crystals with the ice. The method is promising and is to be pursued further at a convenient season.

Ice is commonly assigned to the hexagonal system of crystals[2] and is conveniently referred to a triangular space lattice each cell of which has sides

a and a height c. An elementary triangular cell of such a structure will have a volume $V = \sqrt{3}a^2c/4$. If the density is ρ, the mass of a molecule m and the number of molecules per cell n, the mass of the cell is $nm = 3a^2c\rho/4$. For ice[2] $c/a = 1.4026$, $\rho = .91$ gm/cm^3., $m = 29.73 \times 10^{-24}$ gm. The molecular weight is taken, as the arrangement of diffracting centers is fundamentally that of the molecule, each pair of hydrogen atoms being presumably near an oxygen atom. These values gave

$$a = (54.35n)^{\frac{1}{3}} \times 10^{-8} \text{ cm.}$$

For triangular lattices the following values of n occur: simple lattice, $n = \frac{1}{2}$; two interpenetrating lattices, $n = 1$; three interpenetrating lattices, $n = \frac{3}{4}$ or $\frac{3}{3}$; four interpenetrating lattices, $n = 2$.

Values of a, $a/2$, (the spacing of the $1\bar{2}10$ planes), $a\sqrt{3}/2$ (the spacing of the $10\bar{1}0$ planes and ca (the height of a cell) have been computed and are given in columns 2, 3, 4 and 5 of table 1. When h is the distance of the plate

TABLE 1

1	2	3	4	5	6	7	8	9	10	11
n	$\times 10^{-8}$ cm.				h = 19.00 cm.			h = 14.75 cm.		
	a	$a/2$	$a\frac{1}{2}$	ca	$x_{11\bar{2}0}$	$x_{10\bar{1}0}$	x_{0001}	$x_{11\bar{2}0}$	$x_{10\bar{1}0}$	x_{0001}
$\frac{1}{2}$	3.01	1.50	2.60	4.22	2.67	1.54	0.95	2.07	1.20	0.74
$\frac{3}{4}$	3.44	1.72	2.98	4.83	2.33	2.35	0.83	1.81	1.04	0.65
1	3.79	1.89	3.38	5.32	2.11	1.18	0.76	1.65	0.92	0.59
$\frac{8}{3}$	4.33	2.16	3.75	6.08	1.85	1.07	0.66	1.44	0.83	0.51

from the axis of rotation, x the distance of a given line from the undeviated central image, d the distance between planes in the crystal, λ the wave-length of the radiation and N the order of the spectrum $x/h = N\lambda/d$, giving the relations $d = N\lambda h/x$ and $x = N\lambda h/d$. The wave-length used was the K line of tungsten $\lambda = .211 \times 10^{-8}$ cm. In certain cases h was 19.00 cm., in others 14.75 cm. Values of x corresponding to these values have been calculated for the three fundamental spacings of each of the forms having values of n already givin. They are shown in columns 6 to 11 of table 1. The values of x determined from the four plates used in the calculations and the corresponding values of a are given in tables 2 to 5. The average value of a is 4.74×10^{-8} cm. indicating four interpenetrating lattices. From talb 4 it appears that the 0001 spacing is $c/2$, i.e., the four sets of basal planes occur in pairs. A number of plausible models having such arrangement exist. They may be differentiated by the spacings of the pyramidal planes. It may be shown that in interpenetrating triangular lattices pyramids having indices of the form $(n\bar{O}p)$ have spacings according to the relation

$$d_{n\bar{O}np} = K\frac{a}{2}\sqrt{3}\sin.\phi_{n\bar{O}np},$$

where $\phi_{nO\bar{n}p}$ is the angle between the planes and the basal pinacoid 0001 and K is a factor varying from pyramid to pyramid in the same model and with different models for the same pyramid, its value depending upon the grouping of the planes in a given form. Values of this quantity have been

TABLE 2
$h = 19.00$ cm.

INDEX	x	$d \times 10^8$ cm.	$a \times 10^8$ cm.
$10\bar{1}0$	0.97	4.13	4.77
	1.98 (2)	4.05	4.68
	2.93 (3)	4.11	4.74
$11\bar{2}0$	1.70	2.36	4.72
	3.40 (2)	2.36	4.72
$01\bar{1}0$	0.95	4.22	4.87
	1.92 (2)	4.18	4.82
	2.90 (3)	4.15	4.79
$\bar{2}110$	1.70	2.36	4.72
	3.40 (2)	2.36	4.72
$\bar{1}010$	1.95 (2)	4.12	4.76
Average			4.76±0.04

TABLE 3
$h = 14.75$ cm.

INDEX	x	$d \times 10^8$ cm.	$a \times 10^8$ cm.
$\bar{2}110$	1.34	2.32	4.64
	2.70 (2)	2.31	4.62
$10\bar{1}0$	2.20 (3)	4.25	4.91
	3.00 (4)	4.15	4.79
$\bar{1}100$	1.50 (2)	4.15	4.79
$\bar{2}110$	1.35	2.31	4.62
Average			4.73±0.10

TABLE 4
$h = 14.75$ cm.

INDEX	x	$d \times 10^8$ cm.	$a \times 10^8$ cm.
0001	0.98	3.25	4.78
	1.92 (2)	3.24	4.82
$01\bar{1}0$	1.50 (2)	4.15	4.79
	2.25 (3)	4.15	4.79
$10\bar{1}2$	1.22	2.55	4.68
	2.45 (2)	2.54	4.67
$30\bar{3}4$	2.90	1.07	4.81
$11\bar{2}0$	2.65 (2)	2.34	4.68
$30\bar{3}2$	4.5–5.2	1.19–1.38	
$20\bar{2}1$	4.80 (2)	1.30	4.72
$30\bar{3}1$	4.0–4.5	0.69–0.78	
$40\bar{4}1$	5.5–6.0	1.04–1.13	
Average			4.74±0.06

TABLE 5
$h = 14.75$ cm.

INDEX	x	$d \times 10^8$ cm.	$a \times 10^8$ cm.
0001	1.85 (2)	3.38	4.78
	2.80 (3)	3.34	4.76
$10\bar{1}1$	1.5–2.0	3.1–4.2	
	5.40 (6)	3.46	4.70
$10\bar{1}2$	2.40 (2)	2.58	4.75
$10\bar{1}3$	1.60	1.94	4.72
$30\bar{3}2$	2.50	1.24	4.70
Average			4.72±0.04

determined by inspection of the models and are tabulated in table 6. The data of table 4 are consistent only with the distances calculated for model IV, which is therefore taken to represent the structure. In the tabulations but four models have been considered as these are all that satisfy reasonable considerations of symmetry.

The investigation shows that ice is properly assigned to the hexagonal system, that it consists of four interpenetrating triangular lattices, and that the fundamental spacings are

$$a = 4.74 \times 10^{-8} \text{ cm.}; \quad h = 6.65 \times 10^{-8} \text{ cm.}$$

$d_{11\bar{2}0} = 3.79 \times 10^{-8}$ cm.; $d_{10\bar{1}0} = 2.37 \times 10^{-8}$ cm.; $d_{0001} = 3.32 \times 10^{-8}$ cm.

The arrangement of the lattices is conveniently explained by referring the origin of each lattice to two unit axis making an angle of 120° and a third

TABLE 6

VALUES OF K AND CORRESPONDING APPROXIMATE VALUES OF z WHEN $h = 14.75$ CM.

INDEX	ϕ	Sin ϕ	MODEL I		MODEL II		MODEL III		MODEL IV	
			K	z	K	z	K	z	K	z
$10\bar{1}1$	58°18′30″	0.8509	$\frac{1}{2}$	1.8	1	0.9	1	0.9	1	0.9
$10\bar{1}2$	39° 9′ 9″	0.6293	1	1.2	1	1.2	1	1.2	1	1.2
$10\bar{1}3$	28°22′ 0″	0.4751	$\frac{1}{3}$	3.1	1	1.6	1	1.6	1	1.6
$20\bar{2}1$	72°50′20″	0.9555	$\frac{1}{4}$	3.2	$\frac{1}{2}$	1.6	$\frac{1}{4}$	3.2	$\frac{1}{3}$	2.4
$30\bar{3}1$	78°22′13″	0.9795	$\frac{1}{6}$	4.5	$\frac{1}{3}$	2.2	$\frac{1}{6}$	4.5	$\frac{1}{3}$	4.5
$30\bar{3}2$	67°37′36″	0.9248	$\frac{1}{3}$	2.6	$\frac{1}{3}$	2.6	$\frac{1}{3}$	2.6	$\frac{1}{3}$	2.6
$30\bar{3}4$	50°32′15″	0.7720	$\frac{1}{3}$	2.9	$\frac{1}{3}$	2.9	$\frac{1}{3}$	2.9	$\frac{1}{3}$	2.9
$40\bar{4}1$	81°13′30″	0.9886	$\frac{1}{8}$	6.6	$\frac{1}{3}$	2.2	$\frac{1}{8}$	3.3	$\frac{1}{4}$	3.0

mutually perpendicular to these. The coordinates of the origins are then

$$0,0,0; \ \tfrac{1}{3},\tfrac{2}{3},1/z; \ \tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2}; \ 0,0,\ (z+2)/2z.$$

The values of z, that is the relative displacement of the two planes making up a basal pair, is uncertain and needs further investigation. This requires a careful determination of the relative intensities of the spectra of different orders reflected from the 0001 planes and was beyond the scope of the present investigation. Conditions of symmetry suggest a value of $z = 6$, but this is purely conjectural.

The investigation was pursued under the direction of Prof. A. G. Webster of Clark University in the Physics Laboratory of the Worcester Polytechnic Institute.

[1] *Paris, C.-R. Acad., Sci.*, **157**, Nov. 17, 1913, (924–926).

[2] Dana, *System of Mineralogy*, 1888 ed., p. 205.

FRINGING REEFS OF THE PHILIPPINE ISLANDS.

BY W. M. DAVIS

DEPARTMENT OF GEOLOGY AND GEOGRAPHY, HARVARD UNIVERSITY

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A series of large-scale charts recently published by the United States Coast and Geodetic Survey for certain parts of the Philippine islands are, apart from their value to commerce, of much scientific interest in connection with

the coral-reef problem by reason of the great volume of carefully ascertained facts that they present. They are in various respects more detailed and more accurate than the Admiralty and Hydrographic Charts hitherto available. Many of the islands are thus shown to have more or less minutely embayed shore lines, indicative of the subrecent or recent submergence of an eroded land surface. This is particularly true of Palawan, the southwesternmost member of the group, which has a shore line of most intricate pattern where its western side is indented by Malampaya sound: there can be no question that the coastal features of this kind exhibited on Palawan and many other members of the Philippine group result from the recent partial submergence of an uneven land surface.

According to Darwin's theory of coral reefs, in the form usually presented, shores of submergence should be fronted by barrier reefs; but the Malampaya district of Palawan is not so fronted; its reefs, where they occur, belong to the fringing class, and since Darwin's time fringing reefs have been associated with stationary or emerging shores. Barrier reefs are indeed exceptional in the Philippines, in spite of the repeated occurrence of embayed shore lines on many islands, and the question therefore arises whether the theory of upgrowing reefs on intermittently subsiding foundations is incorrect or incomplete. The object of this paper is to point out that the theory, as ordinarily stated, is incomplete, and that the element needed to complete it is to be found in a seldom-quoted passage from Darwin's own writings, as follows: "If during the prolonged subsidence of a shore, coral-reefs grew for the first time on it, or if an old barrier-reef were destroyed and submerged, and new reefs became attached to the land, these would necessarily at first belong to the fringing class" (Coral Reefs, 124).

This passage may be understood as meaning that the "prolonged subsidence" of an island might be too rapid to permit reef growth, until a pause allowed the establishment of a fringing reef; and also that the rapid subsidence of an island would destroy and submerge a barrier reef previously formed around it during slower subsidence, whereupon a fringing reef would be formed on the new shore line. Thus interpreted, the passage affords a satisfactory explanation of the frequent association of fringing reefs with shore lines of submergence in the Philippines and in certain other archipelagoes, even though such reefs may elsewhere be found on stationary shore lines or on shore lines of emergence. Fringing reefs thus formed may be described as "of a new generation:" they will evidently lie unconformably on the eroded rocks of the shore belt, and their unconformity, as well as the embayments of the shore line, will indicate their association with submergence. The contact of fringing reefs with the marine sediments that ordinarily characterize a non-embayed shore line of emergence would be essentially conformable.

Whether the shores of the Philippines now for the first time have reefs formed upon them, according to the first clause in the above quotation from Darwin, or whether the existing fringes are the successors of "destroyed

and submerged barrier reefs," according to the second clause, may be determined for certain islands by the abundant off-shore soundings. Thus in the case of Palawan, its embayed western coast does not descend rapidly to great depths, but is fronted by a well defined submarine platform, 20 or 30 miles wide, along the seaward edge of which a discontinuous rim rises towards but not to the surface; the rim can be most reasonably explained as an incomplete upgrowth from a barrier reef of an earlier generation on the outer margin of a broad lagoon. It is not possible, in our present ignorance of the geology of Palawan, to determine whether the drowned barrier reef was formed by upgrowth during slow and long continued subsidence, or by outgrowth during a long-enduring still-stand of the island; but in either case, the great breadth of the lagoon plain appears to be the product of a long lasting process, and thus contrasts strongly with the new fringing reefs of the Malampaya district, which are so narrow as to be inconspicuous on charts of the largest scale. Evidently, therefore, the rapid submergence by which the barrier reef of Palawan was drowned must be of recent date.

Similar conclusions may be derived from other parts of the Philippines, where embayed shore lines, relatively narrow fringing reefs, and well defined submarine platforms are frequently found, although no islands are so striking in these respects as Palawan. The platforms cannot be reasonably ascribed to marine abrasion during a higher stand of the islands or a lower stand of the ocean, for the island shores are not clift; and the submergence of the platforms cannot be accounted for by a rise of ocean level, which must everywhere be of the same date, amount and rate, for the platforms vary in depth, and the new fringing reefs vary in breadth. The depth of the Palawan platform for example, increases from 25 or 30 fathoms at its southwestern end to 55 or 60 fathoms near its middle, and then decreases again toward the northeastern end; and the fringing reef, which is hardly chartable near the mid-length of the island where the platform is deep, has a width of 1 or 2 miles at the southwestern end of the island where the platform is relatively shallow.

On the other hand, the northeastern coast of Samar, on the opposite side of the archipelago from Palawan, has a moderately sinuous shore line with delta flats that diminish the initial size of its bays, and fringing reefs that reach forward a mile or so from its points; here the latest submergence cannot be so recent as that of Palawan. But instead of being benched by a submerged platform, the sea bottom off shore from Samar sinks rapidly to a great depth. Moreover, there are long stretches of the coast of Luzon which are neither embayed by arms of the sea, nor enclosed by barrier reefs, nor fronted by submarine platforms: Luzon, unlike many other of the Philippines, has a considerable extent of coastal lowlands, as if the growth of fringing reefs, or the outwash of detritus, or the emergence of the former sea border had increased its low littoral area. Again, Cebú and Negros, which occupy a somewhat central position in the archipelago, are described by Becker as terraced with elevated reefs up to altitudes of 2000 feet or more. The diverse shore features

of the different islands are therefore best explained by local changes of land level, unlike in date, in direction, in amount and in rate.

It is particularly the rapid rate, the recent date, and the considerable amount of subsidence often indicated that are of significance in the coral-reef problem; and this is true not only for the Philippines but also for the other archipelagoes between Asia and Australia. Embayed shore lines indicative of submergence are common though by no means universal in all this region, but well developed barrier reefs are rare. Before the reefs of Cebú and Negros were elevated, and before the platform of Palawan was depressed, barrier reefs must have been more extensive than now in the Philippines, and possibly in the other island groups also; but today no examples of these forms are to be found in the archipelagoes that can compare with the great barrier reefs of northeastern Australia and of New Caledonia; few of the many small islands in the archipelagoes are enclosed by well developed encircling reefs, like those of the Fiji and Society groups; and atolls, which are so striking a feature of the open Pacific, are relatively uncommon in the archipelagoes.

The best explanation of the small development of barrier reefs and atolls in the archipelagoes is to be found, not in the lack of subsidence, which is elsewhere so intimately associated with reef development, for geological and physiographic evidences of subsidence abound on many islands; and surely not in the prevalence of unfavorable conditions as to the temperature and purity of sea water, for fringing reefs flourish; but largely in the occurrence of subsidence of so rapid a rate and in some cases of so great an amount, as to have submerged pre-existing reefs. Moreover the subsidences appear to be in many cases of so recent a date that the new fringing reefs are still narrow; it is presumably for this reason that the drowned barrier reefs and atolls have not yet had time to grow up again to sea level. Added to this is the frequent occurrence of recent uplifts in the archipelagoes, whereby weak marine sediments, overwashed by an abundance of alluvium from rejuvenated rivers, have come to occupy the shore line of certain islands, thus discouraging even the growth of fringing reefs, as around much of the coast of Borneo and along the southern coast of Java; but this aspect of the problem cannot be discussed here.

In view of these facts and inferences the Australasian archipelagoes must be considered much more unstable than the floor of the central Pacific. In that vast region, where reef upgrowth has generally kept pace with changes of level, and where atoll and barrier-reef lagoons have been filled with sediments to a moderate depth, Darwin appears to have been right in concluding that "the subsidence thus counterbalanced must have been slow in an extraordinary degree" (115). Not only movements of depression but modern movements of elevation also have been of moderate measure in the central Pacific, for none of the occasional elevated atolls found there have an altitude of more than a few hundred feet: it is only in Tonga and Fiji that greater measures of modern uplift are recorded, the greatest altitude of a reef in Fiji

being 1030 feet: uplifted reefs in the Australasian archipelagoes are found at altitudes of 2000 and 3000 feet.

On the other hand, the instability of the Australasian region is attested not only by the evidence afforded by coral reefs, but also as above noted, by many geological researches; those by Molengraaff and Abendanon are particularly instructive in this respect, and fully confirm Darwin's statement:—"North of Australia lies the most broken land of the globe, and there the rising parts are surrounded and penetrated by areas of subsidence" (143). Hence while Darwin's general conclusion that "the rate of subsidence has not exceeded the upward growth of corals" (115) seems to hold true for the central Pacific, it is not valid for the region of the archipelagoes. The not infrequent occurrence there of subsidences at a rate sufficient to drown coral reefs ought to satisfy those objectors to Darwin's theory who have urged that it demands too great a uniformity, if not also too slow a rate in the movements of the earth's crust in oceanic areas. But let it be noted that rapid movements of subsidence, and also fringing reefs of a new generation which are formed along a shore where rapid subsidence had drowned pre-existing reefs, were explicitly recognized by Darwin as of possible occurrence—witness the first quotation from his Coral Reefs, above.

Fringing reefs of a new generation on shores of submergence should therefore be accepted as accounted for by Darwin's theory quite as well as other fringing reefs, and indeed as contributing their share towards verifying the theory, even though its author did not recognize any actual fringing reefs as of this origin. He colored all fringing reefs red on his chart, as indicating non-subsiding coasts; for even after pointing out the possibility of their formation where "prolonged subsidence" has taken place, he added: "I have no reason to believe that . . . . any coast has been coloured wrongly with respect to movement indicated" (124). His information about the Philippines was scanty; he colored all their fringing reefs red, as indicative of stationary or rising coasts.

There can indeed be little doubt that a number of other islands also, which have unconformable fringing reefs along their embayed shores, were wrongly classed by Darwin, who took no account of embayments or of unconformities. Striking instances of this kind might be pointed out in the Solomon group, where the small island of Fauro, for example, which is described by Guppy as a deeply denuded volcanic wreck, has a shore line of marked irregularity, a narrow fringing reef, and a well developed submarine platform from 40 to 70 fathoms in depth: such a combination of features proclaims intermittent subsidence after long-continued subaerial erosion, the last movement being more rapid than reef upgrowth; yet all the members of the Solomon group are colored red on Darwin's chart, because the little information he had about them gave "a presumption that they are fringed" (167). Other examples of new fringing reefs on shores of submergence are found on the granitic islands of the Seychelles in the western Indian ocean, as will be again noted below. The

Andaman islands in the Bay of Bengal are elaborately embayed, and a submarine platform several miles in breadth and from 20 to 40 fathoms in depth adjoins them; a bank of similar depth, measuring 35 by 10 miles lies not far away to the east; yet these islands have no barrier reefs and only very narrow and discontinuous fringing reefs; their submergence must be very recent. If the postglacial rise of ocean level were the cause of all these submergences, reefs should everywhere be of about the same volume: as a matter of fact they vary in volume enormously. The fringing reefs of southwest Palawan, of Yap in the western Pacific, and of Rodriquez in the southern Indian ocean are two or three miles wide; various atoll and barrier reefs are half a mile or a mile wide. Others are much narrower, and still others are discontinuous, or so imerfectly developed as not to reach sea level: finally, some submarine banks are flat, without any reef rim. These irregular values speak for variable subsidence of the reef foundations, not for a uniform rise of ocean level.

The occurrence of subsidence in the Australasian region at a more rapid rate than that of coral upgrowth has an interesting bearing on the origin of the numerous submarine banks, of various depths down to 40, 50, or 60 fathoms, in the China sea. It is evident that if islands suffer a movement of subsidence rapid enough to drown their barrier reefs and thus to develop fringing reefs of a new generation, the same subsidence would completely submerge neighboring atolls. Furthermore, if the rapid subsidence of a group of islands were of so recent a date that the resulting fringing reefs are still narrow, the drowned reefs of the submerged atolls might still remain below sea level, even if the amount of submergence had not been great enough to kill all their corals.

Now in view of the proximity of the China sea to the Philippine Islands, it seems reasonable to suppose that its floor has shared some of the many movements by which the islands of the archipelago have been disturbed; hence the submarine banks of that deep basin are best explained, following Darwin's theory, as drowned atolls not yet rebuilt. The date as well as the rate and the amount of a subsidence is therefore of importance in determining whether the atolls that it drowns shall still be submerged. Certainly recent submergence, presumably due to subsidence, has affected the Macclesfield and certain other large submarine banks of the China sea, for corals are growing on the rims of many of them, but have not yet built the rims up to the sea surface. On the other hand in spite of the proximity of the unstable Philippines, the Glacial-control theory explains the Macclesfield bank, which is taken to be a typical example of its kind, as the remains of a huge volcanic cone, originally as large as Hawaii, which stood still long enough to be worn down to small relief in preglacial time, and which, still fixed, was truncated by abrasion at a lower level while the ocean was depressed about 40 fathoms during the Glacial period. So long enduring a stability seems improbable enough even for the central Pacific, and much more improbable for a next-door neighbor of the Philippines.

It may be noted that if the surface forms of the mid-Pacific atolls are considered alone, they can be accounted for very satisfactorily by the Glacial-control theory, which was invented chiefly to explain the special features that they present: but as atolls occur in association with barrier reefs in the Caroline, Society, Fiji and other groups, and as the central islands within the barrier reefs present features which, although perfectly accounted for by Darwin's theory of intermittent subsistence, cannot be accounted for by the Glacial-control theory, its apparent success in explaining atolls is thereby discredited, all the more so in view of the recent discussion by Skeats of the boring in the Funafuti atoll (*Amer. J. Sci., New Haven*, **45**, 1918, 81–90).

The chief value of the ingenious Glacial-control theory may therefore be found not so much in its postulate of the prevalent stability of reef-bearing islands, or in its assumption that reef corals were killed and that reef-bearing islands were abraded while the ocean was chilled and lowered in the Glacial period, but in the emphasis that it gives to changes of sea level from climatic causes as a factor in the coral-reef problem; for it is manifest that if the post-glacial rise of sea level coincide in time with the subsidence of an island, the resulting submergence will be at an accelerated rate and of an increased amount; while if the fall of sea level occasioned by the oncoming of a glacial epoch coincide with a subsidence, the resulting submergence will be at a retarded rate and of a decreased amount. It cannot however be supposed that two processes so unlike in cause as external climatic changes and internal crustal deformations should be closely related in time; their coincidences must be fortuitous. Throughout the central Pacific the rate and amount of recent submergence have not been as a rule too great to be compensated by reef upgrowth; witness the abundant atolls and barrier reefs. But in the region of the Australasian archipelagoes compensation of submergence by reef upgrowth has frequently been unsuccessful; witness the rarity of well developed barrier reefs and the almost entire absence of atolls. As the climatic changes of ocean level must have been everywhere the same, the factors which have determined the success or the failure of reef upgrowth would appear to be the rate, the amount and the date of subsidence.

It may be added that submarine banks, of such form that they are best accounted for as drowned atolls, are rare in the Pacific. A group of ten or more of them is known in an island-free space north of Fiji: several extensive banks also occur in Tonga. Exception must therefore be made in favor of a rapid submergance only for these relatively few examples of submerged Pacific atolls, and the rule that the great majority of Pacific atolls have not been submerged faster than the rate of reef upgrowth is thereby proved. In the Indian ocean, on the other hand, the number of submarine banks bears a larger proportion to that of atolls, and the Indian ocean is generally regarded by geologists as the seat of greater and more recent movements of depression than the Pacific. Recent and rapid subsidences of moderate amounts may therefore be plausibly regarded as of more general occurrence in the Indian

than in the Pacific ocean. The recent date and the rapid rate of subsidence appear to be of greater importance than its amount in the case of the Great Chagos bank, where the submergence does not seem great enough to drown the reef-building corals. Here the muddy central area is 40 or 50 fathoms deep; it is bordered by an irregular sandy bank from one to 5 miles or more in breadth and from 15 to 20 fathoms in depth, on the outer margin of which rises a rim about a mile in width, and only 5 or 10 fathoms in depth; singularly enough, there is little living coral on the outer rim, though knobs of growing coral rise from the central depression. The diameters of the whole mass range from 50 to 75 miles: its form suggests that a prolonged stationary period, during which a broad atoll-reef was developed, was followed by a subsidence of about 10 fathoms, after which a shorter stationary period permitted the upgrowth of a narrower reef; then a rapid and presumably recent subsidence of 5 or more fathoms ensued, since which no effective reef growth has taken place, possibly because, according to Daly's suggestion, the submerged corals were smothered by wave- and current-shifted sediments.

Unfortunately no archipelagoes comparable to those of the Australasian region are present in the Indian ocean to give evidence in the case, but it may be noted that a few high islands which occur in association with the Indian ocean banks—chiefly the granitic islands in the area of the great Seychelles bank—have narrow and unconformable fringing reefs on their deeply eroded and well embayed shores; thus they repeat in a small way the more abundant and therefore more compulsory evidence that is provided by the charts of the Philippines. Further details on these topics are given in an article on "Submarine Banks and the Coral Reef Problem," now in course of publication in the *Journal of Geology*, and in an article on the "Subsidence of Reef-encircled Islands," soon to appear in the *Bulletin of the Geological Society of America*.

DILATION OF THE GREAT ARTERIES DISTAL TO PARTIALLY OCCLUDING BANDS

By William S. Halsted

Medical School, Johns Hopkins University

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The incentive to the work was primarily the desire to cure aneurysms of the abdominal aorta and common iliac arteries.

The method usually employed for the cure of aneurysm is the simplest, viz., the ligation of the affected artery proximal and as close as feasible to the aneurysm. The aorta has been ligated 25 or more times in man, and always with fatal result. Death has been due to hemorrhage or overtaxed heart. Neither gangrene nor paraphlegia has ever resulted from ligation of the aorta

in man. We found, in dogs, as was to have been expected, that fine, completely occluding, ligatures (sizes C or E sewing silk) applied to the thoracic aorta just below the arch would cut through in about two days, and invariably with promptly fatal hemorrhage; whereas coarse ligatures usually made their way through the aorta wall very slowly and without leakage of blood. A connective tissue diaphragm often forms in the wake of these broader threads and the lumen of the vessel may be more or less completely reëstablished.

It occurred to me after much experimentation that occlusion of the aorta to a degree not sufficient fatally to overburden the human heart might effect the cure of an aortic aneurysm. Knotted ligatures we found to be unsuitable, for a desired degree of constriction or obliteration could not be accurately obtained nor could the crushing of the arterial wall be invariably avoided. Tapes of various materials were tested—of cotton, of chromicized intestinal submucosa, of elastic tissue obtained from the aorta, of aponeurotic white fibrous tissue. These were applied in spiral or cuff form. Best suited to the purpose were bands of metal, of aluminum, accurately rolled in cylindrical form by a little instrument of this kind (exhibit). In the use of these metal bands it was impossible to crush the arterial wall, and the desired amount of obturation could be obtained with precision, and also maintained.

The infolded and snugly opposed intimal surfaces under the compressing band have in no instance adhered to each other, and for the reason that the pressure necessary to produce even a very slight reduction in the lumen of the vessel has, in my experience, invariably caused atrophy of its wall. When the occlusion is complete the necrotic arterial wall included in the metal band becomes replaced by a solid cylindrical cord of fibrous tissue, the substitution taking place from the ends.

An interesting incidental observation which we have made in the course of our experiments with the metal band is this; that a dilation of the artery occurs just below a band when the degree of constriction is of the proper amount. This observation apparently explains in a measure the occurrence of aneurysms of the subclavian artery distal to a cervical rib. Analyzing 525 clinical cases of cervical rib we found 106 in which the subclavian artery had been compressed, and that in 21 of these, aneurysm or dilation of this vessel distal to the site of constriction had been noted.

As to the cause of these aneurysms, five of which have come to the knowledge of the collators, there has been varied conjecture. Commentators are, however, agreed that the occurrence of the dilation would have been less incomprehensible to them had it manifested itself on the proximal instead of the distal side of the compression. Attempts have been made to explain the phenomenon, and the following suggestions offered as to its possible cause:

(1) Weakening of the wall of the subclavian artery from erosion by the rib.

(2) Variable or intermittent pulse pressure occasioned by the normal excursions of the arm.

(3) Vasomotor and vasa vasorum disturbances leading to modified nutritional activities in the wall of the artery.

In casting about for an explanation of these aneurysms there constantly obtruded itself the picture of the dilated arterial trunks which, I find from the study of about 400 cases, has occasionally been noted on the cardiac side of arterio-venous fistulae. In our own clinical and experimental cases, dilation of the artery proximal to the fistula has occurred invariably. For this remarkable manifestation, likewise, no satisfactory cause has been assigned. There might, I thought, be a common cause for both—for the dilation of the subclavian artery distal to the cervical rib, and for the dilation, central to the arterio-venous fistula, of the artery concerned in its formation. Hence, for a number of years, in the course of various experiments in partial occlusion of the arteries, I had somewhat in view the possibility of the production, beyond the point of constriction, of a dilation of the artery, analogous to the dilations which have been observed in cases of cervical rib.

Four years ago when after many trials I had altogether despaired of having the hope realized, I was startled, on examining the abdomen of a dog whose aorta had been constricted for about six months to see that each of the branches of trifurcation had become dilated almost to the size of the main aortic trunk.

With this observation as incentive, Dr. Mont Reid and I, the following winter, constricted the abdominal aorta just above its trifurcation, in many dogs and at intervals explored and reexplored the abdominal cavities, but with negative result. Finally, on investigating the abdomen of the last dog we found the hoped-for dilation. The degree of obturation of the aorta was accurately determined on sacrificing the animal, and the following year the experiments were more advantageously repeated because of the data obtained from this case. Now, that we have apparently determined the relative amount of constriction required to give the most pronounced results we are able in almost every instance to produce the dilation.

As regards the cause of the dilation produced experimentally we may I think, conclude that it is not to be found in any of the three factors which have been proposed as responsible for the dilation observed in cases of cervical rib, viz., (1) vasomotor paralysis, (2) trauma and (3) variable blood pressure.

Ad. 1. *Vasomotor Paralysis.* (a) The vasomotor nerves and the vasa vasorum are destroyed by the moderately constricting and totally occluding bands quite as surely as by those which, occluding almost totally, have produced the greatest amount of dilation. (b) Only a portion of the circumference of the subclavian artery is exposed to the pressure of the cervical rib and the scalenus anticus muscle and hence only a fraction of the vasomotor nerves or vasa vasorum could be pressed upon.

Ad. 2. *Trauma.* (a) The dilation is usually fusiform and distal to the rib. (b) Trauma is excluded as a factor in the experimental dilations.

Ad. 3. *Variable Blood Pressure.* (a) Patients suffering from the pressure-

pain of cervical rib rarely make wide excursion movements of the arm. (b) The degree of occlusion is constant in the experimentally constricted vessel.

When an arterial trunk is ligated it becomes occluded to the first proximal and first distal branches and ultimately reduced to a fibrous strand.

From observations which we have made on man and dogs I am quite sure that there may be a remarkable fall in blood pressure in what I have termed 'the dead arterial pocket,' while there is still little if any sign of diminution in the caliber of this portion of the vessel. For example, the right common carotid was ligated by the writer in a case of aneurysm of the external carotid. About 3 months later, in the course of an operation for the excision of the uncured aneurysm, the internal carotid, dead-pocketed between the circle of Willis and the carotid ventricle, was freely exposed for a considerable distance. It had lost its cylindrical form, being flat and tape-like, and, although evidently possessing a considerable lumen, seemed to be empty. When incised, a few drops of blood oozed without pulse from the little cut. The artery was then resected. Its wall was thickened on one side but the lumen was still perhaps three times that of a radial artery. Similar observations I have made twice on the external iliac of the dog after occlusion of this vessel at its origin from the aorta. In the dead pocket between the aorta and the origin of the circumflex iliac and common trunk of the epigastric and obturator arteries the blood pressure must have been almost nil, because from a little slit in the apparently normal arterial wall of the relatively empty external iliac artery the blood escaped very slowly in a tiny, almost pulseless jet about 1 cm. high; whereas, from the femoral artery, below the profunda, the blood spurted normally from a similar knife-prick.

Hence in an artery doomed to obliteration, it would seem that the blood pressure may be lowered before the occlusion process sets in—the lowered pressure being, perhaps, the immediate factor leading to the obliteration.

Can these observations have any bearing upon the explanation of the dilation of the aorta above its trifurcation and of its triad branches in the dog after partial occlusion; of the dilation of the carotid in the human subject which I have observed in one case after partial occlusion of the innominate combined with ligature of the first and third portions of the right subclavian; and of the aneurysm of the third portion of the subclavian in cases of cervical rib?

In 1906 Dr. Richardson and I made the observation that after partial occlusion of the thoracic aorta the maximum pressure may be permanently lowered and the minimum pressure permanently increased distal to the constricting band; and in recent experiments Dr. Reid and I have observed that after constriction of the lower abdominal aorta the diastolic pressure may be so increased and the systolic pressure so lowered as to reduce the pulse pressure by nearly one half. The blood stream in this case, passing with greater velocity and less pressure through the band prevents the obliteration of the artery to the nearest branch, the pocket being not a dead one as it is in the case of

total obliteration. The blood in this pocket beyond the constriction streams, presumably, in whirlpools, somewhat as in the vein and, also, as in the artery in arterio-venous fistula; the thrill, not palpable at first if the occlusion has been nearly complete, later may be perceived with the finger; and the bruit, always audible with the stethoscope, becomes louder as the peripheral arterial resistance increases.

To these factors, then—to the abnormal play of the blood in the relatively, as distinguished from the absolutely dead pocket and to the absence of normal pulse pressure, essential probably to the maintenance of the integrity of the arterial wall, we may have to look for the solution of our problem.

We have completely occluded the aorta just above the trifurcation only in dogs. Usually there has been no distal dilation, and in a previous paper I made the statement that dilation had not been observed below a totally occluding band. Since then, however, a slight degree of dilation, distal to the completely obturated vessel, has taken place in three instances. A dilation of this ventricle-like portion of the aorta between the band and the trifurcation might be expected even in case of complete occlusion, for the anastomosis is very free in this situation and the dead pocket is usually, and perhaps always too short to become obliterated. Lumbar branches may be given off just below, as they are just above the band.

In two instances I have made the following observation in testing, during the life of the animal, for the patency of the aorta under the band. Pressure with the finger immediately above the band shut off the pulse in what we term the ventricle; whereas, pressure with the back of the scalpel-blade, made as close to the band as possible, did not. In these cases there was a patent lumbar artery so close to the proximal edge of the band that pressure by the finger obliterated it, whereas, the knife blade which could be brought to bear on the aortic wall between this little artery and the upper edge of the band did not interrupt the flow in this important asastomotic branch. The contribution of this little artery to the anastomotic bloodstream was sufficient to convert an impalpable into a palpable pulse. A palpable pulse in the ventricle below the band is so invariable, whether the aorta has been completely occluded or not, that the patency of the artery under the band cannot be definitely determined during the life of the animal unless temporary occlusion of it between the band and the nearest lumbar artery obliterates or decidedly influences the pulse in the ventricle. If pressure above the band does not affect the pulse just below it we may conclude that obturation is complete.

Fortunately it occurred to me a few days ago to restudy, with reference to the possibility of finding depicted a dilation of an artery below a ligature, the sketches of surgeons who in bygone years had experimentally ligated the blood-vessels of animals. I was delightfully surprised to find, in the beautifully illustrated volume of Luigi Porta[1] published in 1845 two drawings which portrayed a pronounced dilation of the aorta and its ventricle immediately below

the site of ligation. The ligatures in the two dogs had been applied eight and fifteen months before the death of the animals. There is a great bundle of dilated vessels—the vasa vasis—bridging the gap between the retracted ends of the divided aorta.

Thus three-quarters of a century ago this great, perhaps the greatest surgeon of Italy, furnished irrefutable proof of a remarkable phenomenon which must eventually have interest for the physiologist, the pathologist and the surgeon. Luigi Porta describes the drawing but makes no further comment upon the dilation.

Before the introduction of antiseptic surgery by Lister, thrombosis quite invariably followed ligation of an artery, and it was to the organization of the thrombus that the surgeon looked for the prevention of secondary hemorrhage and for the preservation of the life of the patient. If thrombi formed in these two cases of Porta they must have been eventually absorbed, for the distribution of the dilated vasa vasis proves that the aortic free ends were patulous, and we have further proof of this in the dilation of the aortic ventricle just below the site of the ligation.

In the course of my experiments in partial occlusion of the arteries I have often studied the illustrations, carefully I thought, in Luigi Porta's work, but not until I scanned them with the particular object in view did I discover the dilations so strikingly manifest. I wonder if anyone has ever commented upon or been interested in these two observations of Porta.

In the human subject I have in one instance observed a remarkable dilation of an artery distal to a partially occluding band. In this case an aluminum band was applied to the innominate artery for the cure of a subclavian aneurysm. A few weeks later, the aneurysm being uninfluenced by this procedure, the subclavian artery was ligated both proximal and distal to the sac, and a cure effected. Three years later a quite cylindrical dilation of the right common carotid was observed; and now, twelve years after the application of the band, the common carotid artery is strikingly dilated throughout its entire length. The band on the innominate can be palpated; the blood is coursing through it, and distal to the band is a distinct bruit (exhibit).

Summary.—1. A partially occluded artery (abdominal aorta, innominate, carotid, subclavian) may dilate distal to the site of constriction.

2. The dilation is circumscribed and has been greatest when the lumen of the artery (the aorta) was reduced to one-third or perhaps one-fourth of its original size.

3. When the obturation has been slight in amount dilation has not been observed; of 7 cases of complete obstruction there was a very moderate degree of dilation in 3, and none in 4.

4. Complete or partial occlusion of the thoracic aorta may be followed by dilation central to the point of constriction.

5. Dilation or aneurysm of the subclavian artery has been observed twenty-seven or more times in cases of cervical rib.

6. The dilation of the subclavian is circumscribed, is distal to the point of constriction, and strikingly resembles the dilation which we have produced experimentally.

7. The dilation of the artery proximal to an arterio-venous fistula and distal to a partially occluding band may prove to be referable to the same cause.

8. When the lumen of the aorta is considerably constricted the systolic pressure may be permanently so lowered and the diastolic pressure so increased that the pulse pressure may be diminished by one-half.

9. The experimentally produced dilations and the aneurysms of the subclavian artery in cases of cervical rib are probably not due to vasomotor paralysis, trauma, or sudden variations in blood pressure.

10. The abnormal, whirlpool-like play of the blood in the relatively dead pocket just below the site of the constriction, and the lowered pulse pressure may be the chief factors concerned in the production of the dilation.

11. Bands, rolled ever so tightly, do not rupture the intima.

12. Intimal surfaces, brought, however gently, in contact by bands or ligatures do not, in our experience, unite by first intention, for the force necessary to occlude the artery is sufficient to cause necrosis of the arterial wall.

13. The death of the arterial wall having been brought about by the pressure of the band, a gradual substitution of the necrotic tissue takes place, the new vessels penetrating it from both ends. It is, I believe, in this manner that an artery becomes occluded, and it is thus that a fibrous cord forms within the constricting band.

[1] Luigi Porta. Delle alterazioni patologiche delle arterie per la legatura e la torsione. Milano, 1845, pp. 350, 351, plate V, figs. 3 and 5.

ON THE CORRECTION OF OPTICAL SURFACES

By A. A. Michelson

Ryerson Physical Laboratory, University of Chicago

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In a recent number of the *Philosophical Magazine*, an interesting method for correcting optical surfaces by means of the interferometer, was developed by Mr. Twyman. While nothing in the paper indicates that the method is limited to relatively small surfaces, it would appear that such an application to mirrors and lenses of the size of modern astronomical telescopes can hardly be contemplated as this would involve interferometers of at least equal dimensions.

It was hoped that a modification of Mr. Twyman's method, with an interferometer of usual size, could nevertheless be employed for large lenses or mirrors.

It was found, however, unless the two optical paths of the interferometers were equal, which would involve the presence of a second lens equal to the one to be corrected, that the circular interference bands are extremely small and difficult to observe.

The following simple, fairly direct method, obviates all these difficulties and has given excellent results.

A silt in the focus of the mirror or lens to be tested is illuminated by light from a Nernst glower, concentrated by means of a microscope objective and a total reflection prism. The light returns immediately above the prism forming an image of the slit which is viewed through a microscope with a $\frac{1}{2}$ inch objective.

A series of screens (an adjustable double slit would be much better) with two rectangular apertures are placed in succession in front of the lens or mirror to be tested; one of the apertures being central and the other at varying distances from the center.

The resulting diffraction figure will be a series of bands parallel with the slit, and the position of the central band (achromatic in white light) will remain constant if the adjustment is right and the mirror perfect.

The error in light waves will be half the observed error in fractions of the fringe-width.

The lens or mirror is rotated through the entire circumference, at intervals of 45 degrees or less, and the same operation repeated; and the results plotted on the corresponding chart, which gives accordingly the error in light-waves at every selected point of the surface.

This process applied to a 5-inch achromatic lens showed errors so small that artificial errors were introduced by placing in the path of the pencil a plane parallel plate which had been made roughly cylindrical by retouching locally.

The errors were then measured as described, and amounted to about seven-tenths of a light wave at the greatest. The corrector plate was again retouched by local polishing, and after a half dozen trials (time occupied being about six hours) the errors were reduced to the order of one- or two-hundredths of a light-wave; and the resulting image (which was badly astigmatic) was rendered practically perfect.

It is clear that such a process may be applied to even the largest astronomical mirrors or lenses, both in the original figuring and in the final correction; further, this final correction may be made upon the auxiliary plate, thus incurring no danger to the objective.

With evident modification the method applies to the correction of prisms and gratings. In the latter, however, since the light must be nearly homogeneous, there may not be sufficient intensity to observe the interference fringes under the high magnification required.

It may therefore be of advantage to apply the interferometer (replacing one of the mirrors by the grating) or even more simply, by observing the interference of the light reflected from a plane surface with that diffracted from the grating.

In either case, when the adjustment is perfect, the fringes are concentric circles—which remain constant when the eye or the observing telescope is moved about in any direction, if the grating is perfect; and if not, measurement of the diameters of the circles gives the error.

If, however, the difference of path in the interferometer is small, a curious singularity is presented. The interference fringes are no longer circles but complicated forms expressible by the formula:

$$\Delta = (y - y_0)\ (x^2 + y^2) + \alpha x + \beta y.$$

Further details will appear in a coming number of the *Astrophysical Journal*.